Adventures in Supersingularland: A Look at Supersingular Isogeny Graphs

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This is joint work with Sarah Arpin, Catalina Camacho-Navarro, Kristin Lauter, Joelle Lim, Kristina Nelson and Travis Scholl.

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Protocols:

- SIKE (`https://sike.org`),
- CSIDH (`https://csidh.isogeny.org/`),
- signature schemes (GPS, SeaSign, CSI-FiSh)

Bird's eye view of public key cryptography

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Key exchange after Diffie-Hellman

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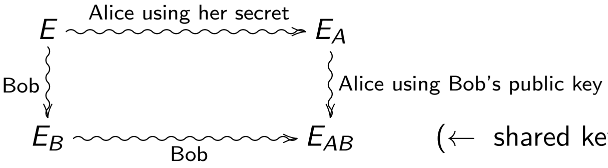

($\leftarrow$ shared key)

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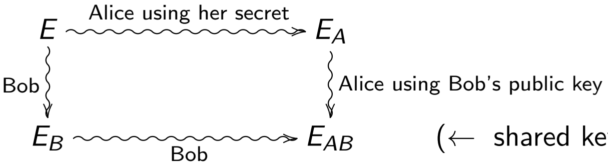

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Isogeny crypto

The arrows correspond to paths in an isogeny graph

What are supersingular isogeny graphs

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2. $\mathbb{F}_{p^2}$, finite field of p^2 elements, quadratic extension of $\mathbb{F}_p$,
3. $\overline{\mathbb{F}}_p$: It suffices to think that everything is defined over a finite field $\mathbb{F}_q$ with $q = p^n$ for some n.

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$$E : y^2 = x^3 + ax + b \quad a, b \in \mathbb{F}_{p^2}$$

together with a point at infinity ∞ and such that

$$|\{(x, y) : x, y \in \mathbb{F}_{p^2} \text{ and } y^2 = x^3 = ax + b\} \cup \{\infty\}| = (p + 1)^2$$

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If $a, b \in \mathbb{F}_p$, then we say that E is defined over $\mathbb{F}_p$.

Labels of vertices

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3. edges: isogenies of elliptic curves, *up to equivalence, up to dual isogenies, of a certain degree*

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Isogenies in an example

Isogenies are maps of elliptic curves, given by rational maps:

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There are two points mapping to ∞, so we say that this isogeny has degree 2.

Supersingular isogeny graphs

Fix a prime p (big) and a prime ℓ (small).

The supersingular isogeny graph $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$

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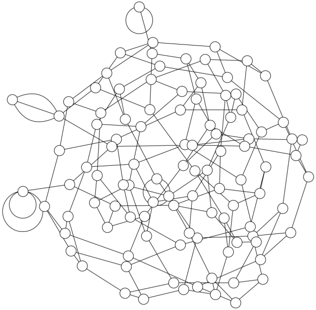

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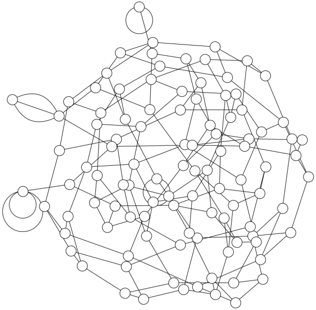
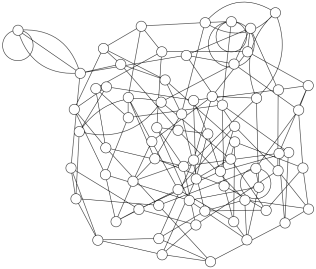


$p = 827$ and $\ell = 3$



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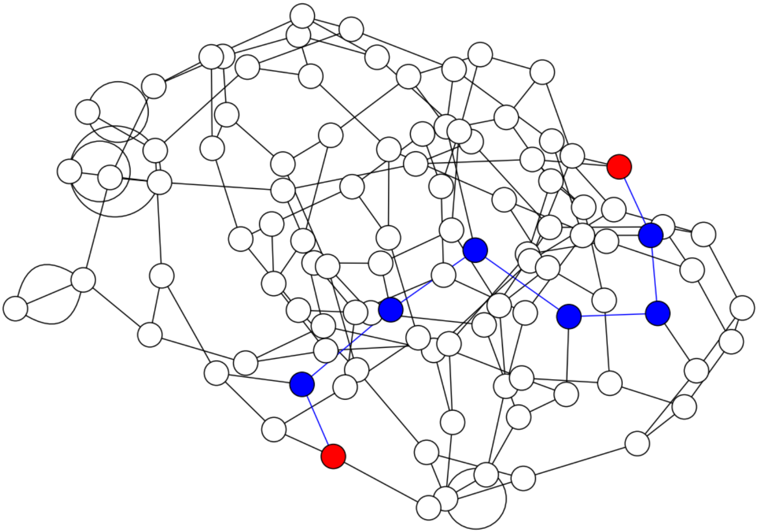
Path finding is hard

For $p = 1223$ and $\ell = 2$, shortest path between two random vertices:



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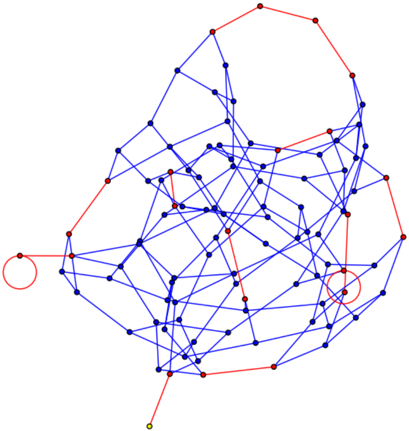

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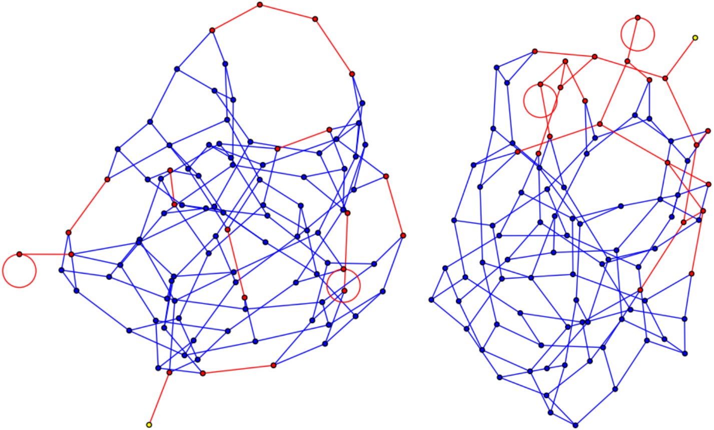

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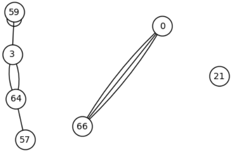
The spine for $\ell = 2$



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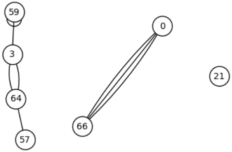

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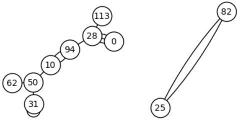

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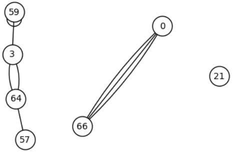

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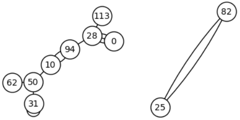

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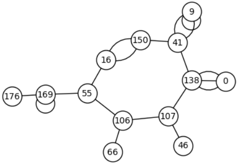
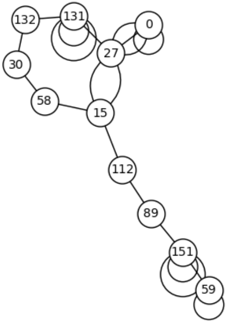


$p = 191$

Examples of the spine

The spine for $\ell = 3$



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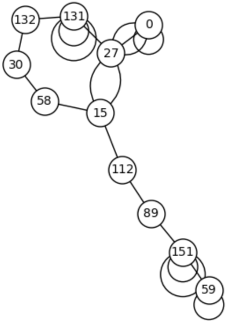

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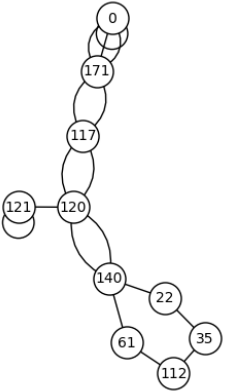

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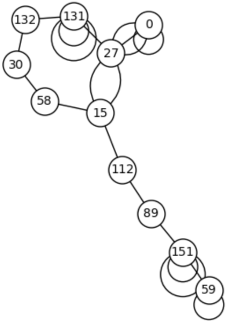

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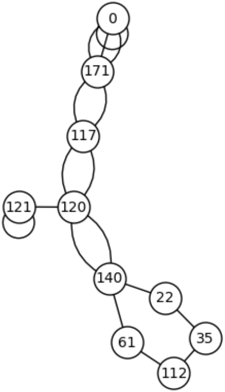

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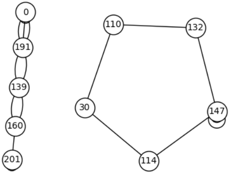

$p = 227$

Visible structure

In the last picture, we see the nice cycle with 5 vertices and another component also with 5 vertices.

Everything over $\mathbb{F}_p$: the graph $\mathcal{G}_\ell(\mathbb{F}_p)$

Fix ℓ a small prime and p a large prime.

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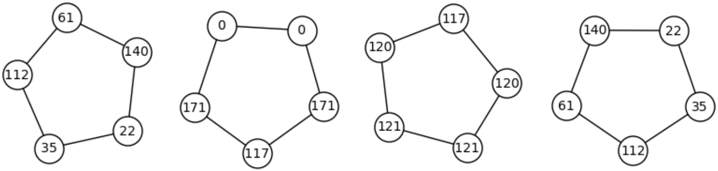
Example with $p = 179$ and $\ell = 3$

labels = j-invariants of the curves



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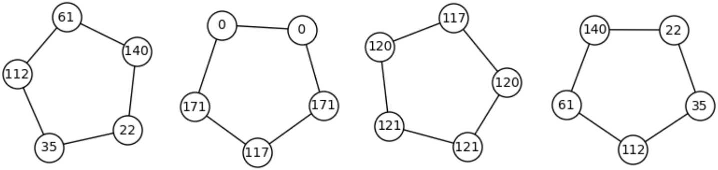

Any ℓ-isogeny graph $\mathcal{G}_\ell(\mathbb{F}_p)$ for $\ell > 2$ will be a union of cycles.

2-Isogenies: the graph $\mathcal{G}_2(\mathbb{F}_p)$

It depends on p mod 8:

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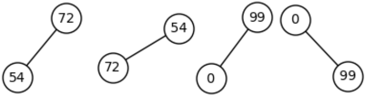

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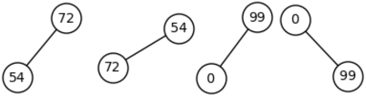
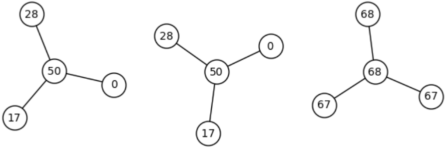


2. $p \equiv 3 \bmod 8$: claws



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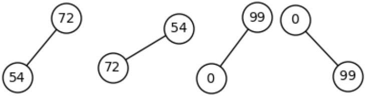
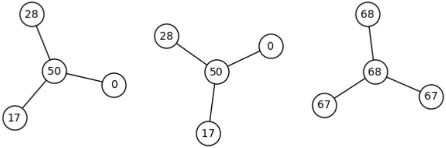
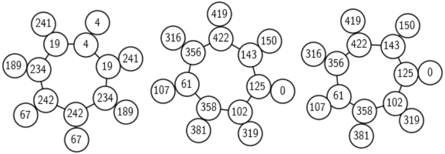


2. $p \equiv 3 \bmod 8$: claws



3. $p \equiv 7 \bmod 8$: volcanoes

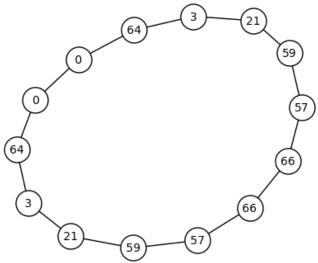


How to pass from $\mathcal{G}_\ell(\mathbb{F}_p)$ to the Spine $\mathcal{S}$

Two-step process

1. Identify vertices with the same j-invariant,
2. add edges that were not defined over $\mathbb{F}_p$.

For $\ell = 3$ and $p = 101$



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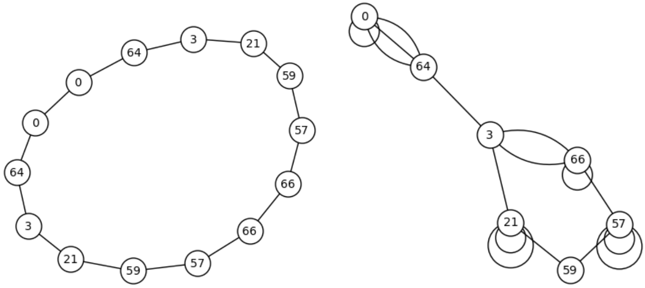

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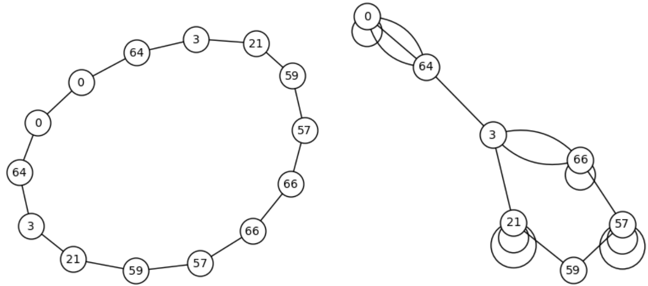
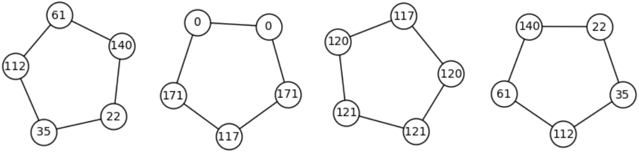


Lemma

Whenever we add an edge that does not correspond to an isogeny defined over $\mathbb{F}_p$, we get a double edge.

Neighbours

$\mathcal{G}_\ell(\mathbb{F}_p)$ for $p = 179, \ell = 3$



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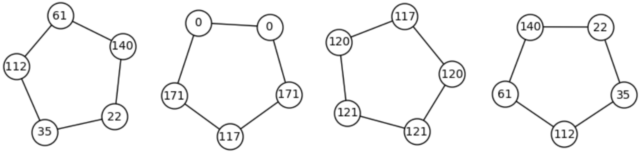
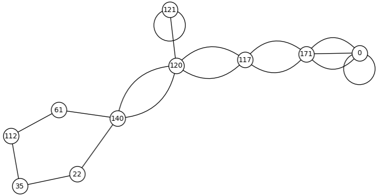


The Neighbour Lemma

Whenever the two vertices in $\mathcal{G}_\ell(\mathbb{F}_p)$ with j-invariant a do not have the same neighbours, there is a double edge from a in $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$.



Double edges using modular polynomials

Proposition

There exists a polynomial $\mathrm{Res}_\ell(X)$ of degree bounded by $2\ell(2\ell-1)$ such that there is a double edge from j in $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$ if and only if $\mathrm{Res}_\ell(j) = 0$.

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$\ell = 2$

There is a double edge from j if and only if it is a root of

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This is a product of Hilbert class polynomials: whether the roots exist or don't depends on a congruence class of p!

1. $p \equiv 1 \bmod 4$ then $j = 1728$ is not a supersingular j-invariant,
2. $p \equiv 1 \bmod 3$ then $j = 0$ is not a supersingular j-invariant, ...

Works the same for any ℓ.

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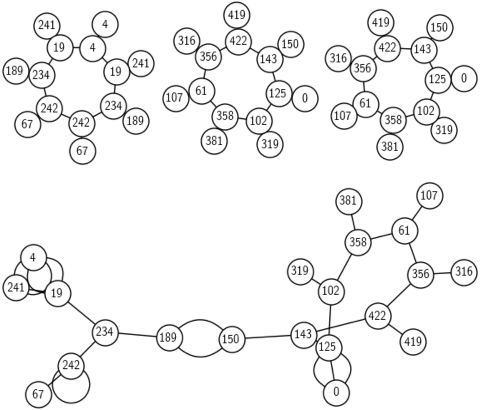
3. At most one pair of vertices admits a new double edge. (*attaching.*)

Example for $\ell = 2$ and $p = 431$

Example

The graph above is $\mathcal{G}_2(\mathbb{F}_p)$ and the graph below is the spine in $\mathcal{G}_2(\overline{\mathbb{F}}_p)$.

We have
$1728 \bmod 431 = 4$
$8000 \bmod 431 = 242$
and 189 and 150 are the two roots of the polynomial $(X^2 + 191025X - 121287375)$ that we saw as a factor of $\mathrm{Res}_2(X)$.



Summary of what the Spine looks like for $\ell = 2$

The $\mathbb{F}_p$-subgraph $\mathcal{S} \subset \mathcal{G}_2(\overline{\mathbb{F}}_p)$:

1. for $p \equiv 1 \bmod 4$, we see single edges, with a possible vertex with a loop at $j = 8000$ and one possible component of size 4,
2. for $p \equiv 3 \bmod 8$, we see claws, with one claw collapsed to an edge ($j = 1728$), and a possible pair of claws joined by a double edge,
3. for $p \equiv 7 \bmod 8$, we see volcanoes, one of the volcanoes will be collapsed and possibly two volcanoes will get attached by a double edge to form a large component.

Why we call the Spine the Spine:

The finite field $\mathbb{F}_{p^2}$ has an involution: every j is sent to

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How accurate is this picture?

How central is the Spine?

Opposite vertices

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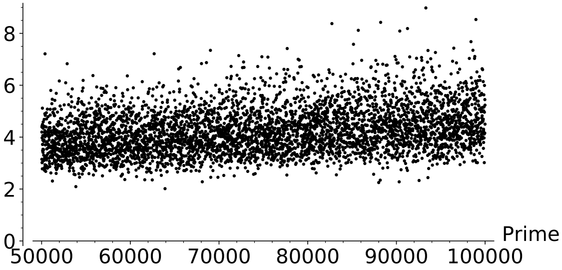
We say that j and j' are opposite vertices if the shortest path between j and j' passes through the spine.

Ratios of the number of opposite conjugate pairs (pairs j, j^p) to opposite arbitrary pairs. $(\ell = 2)$



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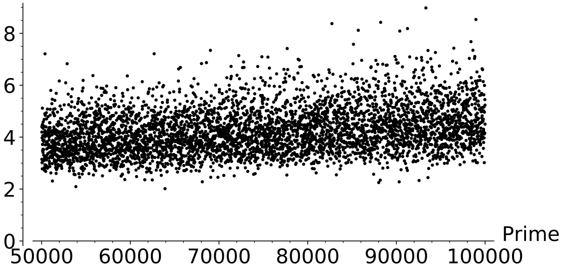

Conclusion: the spine is not in the *middle* of the graph.

Distances to the spine

Since the spine is a subgraph of size $\approx \sqrt{p}$, folklore is that we will reach the spine in approximately $\frac{1}{2}\log(p)$ steps.

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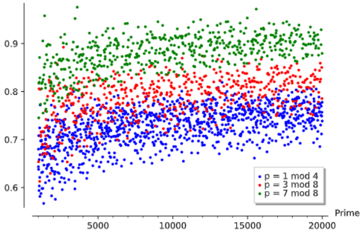

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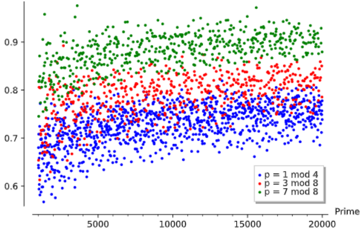
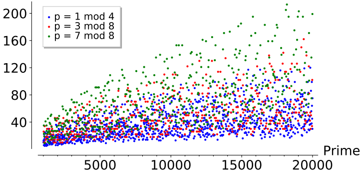




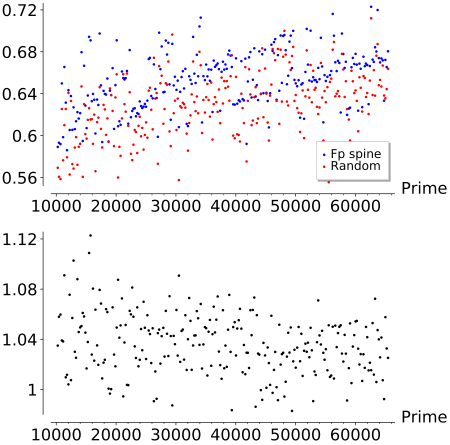
d_p for primes p mod 8

size of $\mathcal{S}$ for p mod 8.

Distances of components for $\ell = 2$ and $p \equiv 1 \bmod 4$

Normalized (by the diameter) distance between $\mathcal{S}$ components (blue) and random pairs (red)

Ratio of distances between components and the distances between random pairs



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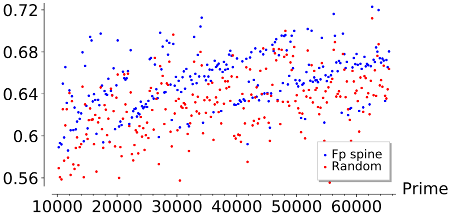

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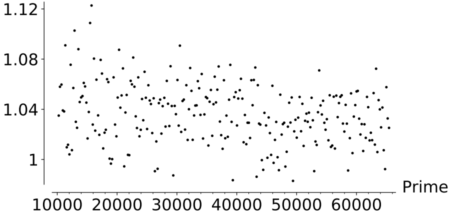

Why is the distance of the edges between $\mathbb{F}_p$ vertices larger than the distance of random vertices?

Diameter of $\mathcal{G}_2(\overline{\mathbb{F}_p})$

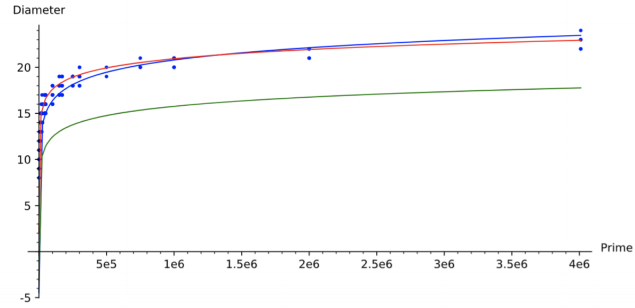

Figure 6.1: *Diameters of 2-isogeny graph over* $\overline{\mathbb{F}}_p$*, with* $y = \log_2(p/12) + \log_2(12) + 1$ *(red) and* $y = \frac{4}{3}\log_2(p/12) - 1$ *(blue).*

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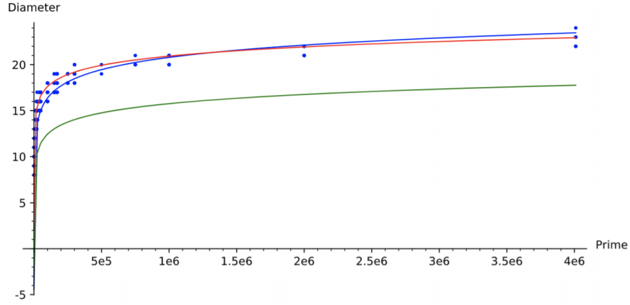

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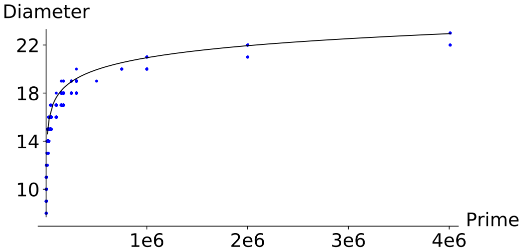
- Blue line: similar to LPS graphs (Lubotzky-Phillips-Sarnak)
- Red line: similar to random Ramanujan graphs

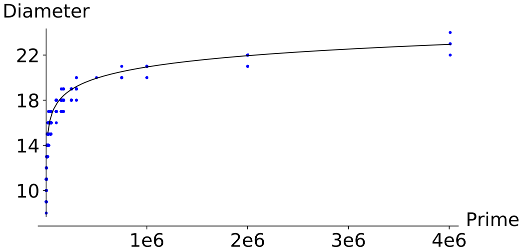
Diameters modulo 12

The line is $y = \log_2(p/12) + \log_2(12) + 1$.

$p \equiv$
$1, 7 \bmod 12$



$p \equiv$
$5, 11 \bmod 12$



Not just a picture

average diameter for $100,000 < p < 300,000$			
1 mod 12	17.2190476190476	5 mod 12	17.8761061946903
7 mod 12	17.7346938775510	11 mod 12	17.9919354838710
average diameter for $300,000 < p < 500,000$			
1 mod 12	18.4000000000000	5 mod 12	18.9230769230769
7 mod 12	18.8235294117647	11 mod 12	19.1000000000000

Average diameters sorted by primes modulo 12. The first data set contains around 100 primes in each congruence class, the latter between 10 to 17 primes.

Trends Modulo 12

For $p \equiv 1 \pmod{12}$:

- smaller 2-isogeny graph diameters,
- spine as disconnected as possible,
- fewer vertices in the spine.

For $p \equiv 11 \pmod{12}$:

- larger 2-isogeny graph diameters,
- fewer (but larger) connected components in the spine,
- more vertices in the spine.

Thank you for your attention!

For more, go to: eprint 2019/1056